Coulomb’s Law and Electrostatics

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Electric charge
Types of charges

Many elementary particles (protons, electrons) carry charge.

Charge can be either: positive (+) or: negative (-).

An object has net charge if its (+) and (-) charge contents differ.

+ +, - - (like charges) repel each other.

+ - (opposite charges) attract each other.
Coulomb’s Law

Experiments in the 1700s showed that charges act on each other with a force $f$ of magnitude:
- inversely proportional with the square of the distance between them, $r^2$.
- directly proportional with the magnitude of both charges, $q_1$ and $q_2$.

\[
f = \kappa \frac{q_1 q_2}{r^2}\]

where \( \kappa = \frac{1}{4\pi\varepsilon_0} \)

The unit of charge is: Coulomb (C).

\( \varepsilon_0 = 8.85 \times 10^{-12} \text{F/m} \) is the relative permittivity of vacuum.
Electrostatic interaction energy

Assume that we have two charges, $q_1$ and $q_2$.

Their interaction energy is the work gained/lost by bringing one charge from $\infty$ to its current position:

\[
 u(r_{12}) = -\int_{\infty}^{r_{12}} f(r) dr = -\int_{\infty}^{r_{12}} \frac{1}{4\pi \varepsilon_0} \frac{q_1 q_2}{r^2} dr
\]

\[
 u(r_{12}) = -\frac{q_1 q_2}{4\pi \varepsilon_0} \int_{\infty}^{r_{12}} \frac{dr}{r^2} = -\frac{q_1 q_2}{4\pi \varepsilon_0 r_{12}}
\]

Overall, the electrostatic interaction energy drops with the distance $r$: 

\[
 u(r) = \frac{1}{4\pi \varepsilon_0} \frac{q_1 q_2}{r}
\]
Electrostatic interactions can hold charges together.

Imagine that we have 3 charges as indicated below. What makes this system stable? Their total interaction energy is:

\[ u = u_{1,2} + u_{1,3} + u_{2,3} \]

\[ u = \kappa \frac{q_1 q_2}{r_{1,2}} + \kappa \frac{q_1 q_3}{r_{1,3}} + \kappa \frac{q_2 q_3}{r_{2,3}} = -2 \frac{\kappa q}{r} + \frac{\kappa}{2r} = \frac{\kappa(1 - 4q)}{2r} \]

- \( q > 1/4 \) \( \Rightarrow \) \( u < 0 \) \( \Rightarrow \) The system is energetically stable
- \( q < 1/4 \) \( \Rightarrow \) \( u > 0 \) \( \Rightarrow \) The system is energetically unstable

Will this system stay like this?
Electrostatic interactions hold materials together.

Unlike short-range bonds, electrostatic interactions act over long distances. They hold crystal structures together. The electrostatic interaction energy of a NaCl crystal structure for 1 atom... Starting with Row 1:

\[ u_{\text{Row}_1} = -\frac{2\kappa e^2}{a} + \frac{2\kappa e^2}{2a} - \frac{2\kappa e^2}{3a} + \ldots = -\frac{2\kappa e^2}{a} \ln(2) = -1.386 \frac{\kappa e^2}{a} \]

\[ U_{\text{atom}} = \sum_{i} u_{\text{Row}_i} = -1.747 \frac{\kappa e^2}{a} \]
Charge interactions are weaker in media

Some molecules can polarize (their positive and negative charges shift rel. to each other).

The + charge will dominate on one side of the molecule, – charge will dominate on the other.

The reorientation of molecular dipoles modifies the force and interaction energy.

Would the forces be larger or smaller without the dielectric?
Toy problem: Two charges and a molecular dipole

![Diagram of two charges and a molecular dipole]

Electrostatic interaction energy without the dipole:

\[ u_0 = -\frac{\kappa Q^2}{R} \]

With the dipole:

\[ u = -\frac{\kappa Q^2}{R} + 4 \frac{\kappa q Q}{R + r} - 4 \frac{\kappa q Q}{R - r} + \frac{\kappa q^2 (r - r_0)}{r r_0} \]

Energy change due to the dipole:

\[ \Delta u = -8 \frac{\kappa rqQ}{R^2 - r^2} + \frac{\kappa q^2 (r - r_0)}{rr_0} \approx \frac{\kappa q^2 (r - r_0)R^2}{R^2 rr_0} - 8\kappa r^2 r_0 qQ \]

The dipole weakens the interaction energy if:

\[ \frac{R^2}{r^2} \left( \frac{r}{r_0} - 1 \right) > 8 \frac{Q}{q} \]
Effects of dipole weaken with temperature

The dipole weakens the interaction energy between $Q$ and $-Q$ if:

$$\frac{R^2}{r^2} > 8 \frac{Q}{q}$$

Ballpark numbers, water

$$\left( \frac{0.5 \text{m}}{5 \times 10^{-10} \text{m}} \right)^2 > 8 \frac{Q}{1.6 \times 10^{-19} \text{C}} \Rightarrow Q < 0.02 \text{C}$$

The internal energy of the dipole ($Z$ is the partition function):

$$U = -\frac{1}{Z} \frac{\partial Z}{\partial \beta} = -8 \frac{\kappa rqQ}{R^2} \left[ e \frac{8 \beta \kappa rqQ}{R^2} - e \frac{-8 \beta \kappa rqQ}{R^2} \right]$$

$$U = -8 \frac{\kappa rqQ}{R^2}, \text{ if } T \rightarrow 0$$

$$U = 0, \text{ if } T \rightarrow \infty$$

This means that the effects of dielectrics decrease with temperature as dipoles orient randomly.
Charge interactions are weaker in media

In polarizable media, the potential becomes:

\[ u(r) = \frac{1}{4\pi\varepsilon_0} \frac{q_1 q_2}{Dr} \]

\( D \) is the dielectric constant.
The more polarizable the medium, the larger \( D \).

Molecules could be:
- permanently polarized
- polarized when other charges are present.

Polarity of water molecules shield ionic interactions.
This is why salts ionize in water.
The Bjerrum length

As seen in the dipole model, the partition function contains terms of the form: \( \frac{u}{kT} \)

When does \( kT \) dominate over \( u \)?

**Bjerrum length**: distance where Coulomb energy = thermal energy for elementary charges.

\[
\begin{align*}
u &= \frac{1}{4\pi\varepsilon_0 Dl_B} e^2 = k_B T \\
l_B &= \frac{\kappa e^2}{Dk_B T} = \frac{\kappa N_A e^2}{DRT}
\end{align*}
\]

The Bjerrum length for two electrons:

\[
l_B = \frac{(1.6 \times 10^{-19})^2}{4\pi 8.85 \times 10^{-12} \times 1.38 \times 10^{-23} \times 273}
\]

\[
l_B = 6.11 \times 10^{-8} \text{ m} = 611 \text{ Å}
\]
Electrostatic forces as vectors

The electrostatic force acts along the line connecting two charges. Electrostatic forces add together as vectors. Vectors have magnitude and direction. Scalars have only magnitude.

\[ f = \frac{1}{4\pi\varepsilon_0} \frac{q_1 q_2}{Dr^2} \]

The components of a vector are its projections on the axes of a coordinate system.

\[ f_x = f \cos(\alpha) \]
\[ f_y = f \sin(\alpha) \]

Other vectors:
- velocity
- torque
- surface vector / vector area
Vector addition

Add vectors = add their components, which become components of the vector sum.

\[ f_{\text{sum},x} = f_{1,x} + f_{2,x} = f_1 \cos(\alpha_1) + f_2 \cos(\alpha_2) \]

\[ f_{\text{sum},y} = f_{1,y} + f_{2,y} = f_1 \sin(\alpha_1) + f_2 \sin(\alpha_2) \]

The graphical method is to draw parallels with each vector.

The diagonal of the parallelogram is the vector sum.
The electrostatic field

The electrostatic field characterizes the space where charges exist.

At any point in space, it is defined as the force that would act on a unit (+1C) test charge.

\[
E(r) = \frac{1}{4\pi\varepsilon_0} \frac{q_1}{D r_1^2} \frac{r_1}{r_1} + \frac{1}{4\pi\varepsilon_0} \frac{q_2}{D r_2^2} \frac{r_2}{r_2} + \ldots \quad \text{(sum over all charges)}
\]
The electric field of a positive charge

The electric field of a positive point charge points radially away from the charge. It has spherical symmetry.

\[
E(r) = \frac{q}{4\pi\varepsilon_0 Dr^2} \frac{r}{r}
\]

Garden hose analogy...
The electric field of a positive charge

The electric field of a negative point charge points radially towards the charge. It has spherical symmetry.

Garden hose analogy...

\[ E(r) = -\frac{q}{4\pi\varepsilon_0 D r^2} \frac{r}{r} \]
The electric field of a dipole

The electric field of a dipole has cylindrical symmetry.

Why do like charges repel? Why do opposite charges attract?...

http://www.ribbonfarm.com/2015/06/23/where-do-electric-forces-come-from/
The electric field between two planar electrodes

The electric field between two parallel planar electrodes is linear.

Lettuce seeds floating in vegetable oil
The scalar (dot) product of two vectors

Dot product = the product of two vectors’ magnitudes with the cosine of their angle.

Example:

the amount of fluid flowing through a surface element per unit time...
... is proportional to dot product of velocity and area vectors.

\[ \vec{A} \cdot \vec{B} = |\vec{A}| \cdot |\vec{B}| \cdot \cos \theta \]
The flux of the electric field

The flux of the electric field through a given surface is the following integral:

\[
\Phi = \Phi = \int_{\text{surface}} D \mathbf{E} \, ds = \frac{1}{4\pi \varepsilon_0} \int_{\text{surface}} \frac{q \mathbf{r}}{r^2} \, ds
\]

Example:
For a sphere with a point charge at its center:

\[
\Phi = \frac{1}{4\pi \varepsilon_0} \frac{q}{R^2} \int_{\text{surface}} \frac{\mathbf{r}}{r^2} \, ds = \frac{1}{4\pi \varepsilon_0} \frac{q}{R^2} 4\pi R^2
\]

\[
\Phi = 4\pi \frac{1}{4\pi \varepsilon_0} q = \frac{q}{\varepsilon_0}
\]

The flux is independent of the radius of the sphere.

This then implies a proportionality for any spherical sections (e.g., flux through all hemispheres ~ half of charge inside).
The flux is independent of the balloon’s shape

Imagine an outer balloon surrounding the inner sphere and the charge. For a given surface element:

\[ \Phi_{\text{outer}} = D E(R) S(R) \cos \theta \]

\[ E(R) = \frac{1}{4 \pi \varepsilon_0} \frac{q}{D R^2} = E(r) \frac{r^2}{R^2} \]

\[ \Phi_{\text{outer}} = D E(r) \frac{r^2}{R^2} S(R) \cos \theta = \Phi_{\text{inner}} \]

For any surrounding surface: \[ \Phi = \frac{q}{\varepsilon_0} \]
Gauss’ Law

For multiple charges surrounded by a surface we obtain Gauss’ Law:

\[
\Phi = \int D E d s = \frac{1}{4 \pi \varepsilon_0} \int D (E_1 + E_2 + \ldots) d s = \frac{1}{\varepsilon_0} \sum_i q_i
\]

Gauss’ Law:
Flux through a closed surface is proportional with the sum of charges in its interior.

For charge that is continuously distributed with charge density \( \rho(x,y,z) \):

\[
\Phi = \int_{\text{surface}} D E d s = \frac{1}{\varepsilon_0} \int V \rho d V
\]