Diffusion.
Chemical rate models.
Mass-action kinetics.

Monday, September 25, 2017
BME/CHE/PHY 558, Physical & Quantitative Biology
Rutgers University: Chemical Thermodynamics
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Challenge: Fixing the vapor pressure

Saltwater has less vapor pressure than freshwater.

Restore vapor pressure to its freshwater level!

\[
p_{B,\text{fresh}} = p_{B,\text{int}} e^{\frac{z w_{BB}}{2RT_0}}
\]

\[
p_{B,\text{salty}} \approx x_B p_{B,\text{int}} e^{\frac{z w_{BB}}{2RT_0}}
\]

Freshwater, \( B \)

Saltwater (Ideal solution)
Solution: Fixing the vapor pressure

Saltwater has less vapor pressure than freshwater.

Restore vapor pressure to the freshwater level!

\[ p_{B,\text{fresh}} = p_{B,\text{int}} \frac{z w_{BB}^{\text{B}}}{2 R T_0} \]

\[ p_{B,\text{salty}} \approx x_B p_{B,\text{int}} e^{\frac{z w_{BB}^{\text{B}}}{2 R T_1}} \]

Raise the temperature to \( T_1 \)!

\[ p_{B,\text{salty}}(T_1) = p_{B,\text{fresh}}(T_0) \]

\[ e^{\frac{z w_{BB}^{\text{B}} + \ln(x_B)}{2 R T_1}} = e^{\frac{z w_{BB}^{\text{B}}}{2 R T_0}} \]

\[ \ln(x_B) = \frac{z w_{BB}^{\text{B}}}{2 R} \left( \frac{1}{T_0} - \frac{1}{T_1} \right) \]
Solute elevates boiling temperature of solvent

For example: saltwater boils at higher temperature than freshwater.

Why?  
1) freshwater boils at $T_{b,0} = 100^\circ C$ where $p_{b,\text{fresh}} = 1$ atm.  
2) salt reduces boiling vapor pressure of water.  
3) to boil saltwater, adjust the temperature to: $T_{b,1} > T_{b,0}$.

\[
\ln(x_B) = \frac{ZW_{BB}}{2R} \left( \frac{1}{T_{b,0}} - \frac{1}{T_{b,1}} \right)
\]

\[
\ln(x_B) \approx \frac{ZW_{BB}}{2R} \frac{\Delta T}{T_{b,0}^2}
\]

\[
\Delta T_b = -\frac{2}{ZW_{BB}} RT_{b,0}^2 x_A = \frac{RT_{b,0}^2 x_A}{\Delta h_{vap}}
\]

Boiling temperature of saltwater goes up compared to freshwater.
Solute lowers freezing temperature of solvent

For example, saltwater freezes at lower temperature than pure water.

Why?
1) freshwater freezes at 0°C.
2) salt reduces the escape of water molecules from liquid.
3) for ice to still form, temperature must drop.

Freezing temperature drops with solute concentration $x_A$:

$$\Delta T_f = \frac{RT_f^2 x_A}{\Delta h_{fus}^0} < 0$$
Summary + Application

4 months from now, somewhere in NY 😊
Adding solute causes osmotic pressure

Assume having pure water ($B$) on one side of a semipermeable membrane.

The membrane allows only $B$ molecules (not $A$) to pass through.

$B$ molecules will tend to migrate towards the mixture due to MaxEnt.

How much pressure could halt the flow of $B$ into the mixture?

At equilibrium, the chemical potentials should be equal:

$$\mu_B(p, 1) = \mu_B(p + \pi, x_B)$$
Calculating the osmotic pressure

Start with pure solvent B on both sides. Assume the following process:

1) increase pressure on the mixture side from $p$ to $p + \pi$
2) add solute to the mixture side until solvent’s fraction is $x_B$

1) $\mu_B(p + \pi, 1) = \mu_B(p, 1) + \int_p^{p+\pi} \frac{\partial \mu_B}{\partial p} dp$

2) $\mu_B(p + \pi, x_B) = \mu_B(p, 1) + \pi \nu_B + RT \ln(\gamma_B x_B)$

Maxwell relation:

$$\frac{\partial \mu_B}{\partial p} = \frac{\partial V}{\partial N_B} = v_B$$

$\nu_B(1-x_B)^2$:

$x_B \approx 1 \Rightarrow \gamma_B \approx 1$

$\gamma_B x_B \approx 1 - x_A$

$$\pi = -\frac{RT \ln(\gamma_B x_B)}{v_B} \approx \frac{RT x_A}{v_B}$$
Why are microbes spherical or cylindrical?

- Yeast
- *Escherichia coli* bacterium
Diffusion
Flux

Movement in fluids (liquids or gases) can be characterized by the flux.

Definition of concentration: \( c = \frac{dN}{dV} \)  

Amount of material: \( \Delta N \approx c \Delta V = cA \Delta x \)

Flux = Material per unit time per unit area: \( J = \frac{\Delta N}{A \Delta t} = c \frac{\Delta x}{\Delta t} = cv \)

Flux of Niagara Falls: 2500 liters/second/m²
Force-induced flux

If a force causes the movement, the motion of particles is overdamped. Their speed is proportional to force $f$. (While acceleration $= 0$!)

Then the flux is also proportional to the force:

$$J = cv = rac{c}{\xi} f = Lf$$
Gradient-induced flux: Fick’s Laws

MaxEnt eliminates concentration differences. Concentration gradients cause flux when this happens.

**Fick’s 1\textsuperscript{st} Law:** The flux is proportional to the gradient.

\[ J = -D \frac{dc}{dx} \quad \text{or} \quad \mathbf{J} = -D \nabla c \]

**Fick’s 2\textsuperscript{nd} Law:** The diffusion equation.

\[ \frac{dc}{dt} = -\frac{\partial J}{\partial x} \]

\[ \frac{dc}{dt} = \frac{\partial}{\partial x} \left( D \frac{\partial c}{\partial x} \right) = D \frac{\partial^2 c}{\partial x^2} \quad \text{or} \quad \frac{dc}{dt} = D \nabla^2 c \]
Diffusion through a membrane

Consider a membrane separating reservoirs with concentrations $c_L$ and $c_R$. What is the concentration inside the membrane? After sufficient time, steady state is established.

\[
\frac{dc}{dt} = D \frac{\partial^2 c}{\partial x^2} = 0
\]

Integrate once

\[
\frac{\partial c}{\partial x} = \int 0 \, dx = A_1
\]

Integrate again

\[
c(x) = \int A_1 \, dx = A_1 x + A_2
\]

Boundary conditions:

\[
c(0) = A_1 0 + A_2 = Kc_L
\]
\[
c(h) = A_1 h + c_L = c_R \quad \Rightarrow A_1 = K \frac{c_R - c_L}{h}
\]
\[
c(x) = K \left[ \frac{(c_R - c_L)x}{h} + c_L \right]
\]

partition coeff. permeability

\[
J = -D \frac{dc}{dx} = KD \frac{c_L - c_R}{h} = -P \Delta c
\]
Diffusion in 3 dimensions

Consider a "sticky" sphere towards which molecules diffuse and then disintegrate. Concentration=? Solve the 3D diffusion equation in spherical coordinates at steady state:

\[
\nabla^2 c = \frac{1}{r} \frac{\partial^2 (rc)}{\partial r^2} = 0
\]

Integrate once

\[
\frac{\partial (rc)}{\partial r} = A_1
\]

Integrate again

\[
c(r) = A_1 + \frac{A_2}{r}
\]

nonlinear concentration gradient
Now we use the boundary condition to find the actual solution.

If the concentration at the surface of the sphere is 0, then:

\[
0 = c(a) = A_1 + \frac{A_2}{a} \quad \Rightarrow \quad A_1 = -\frac{A_2}{a}
\]

If the sphere is immersed in a particle bath, \( C_{\infty} \):

\[
\lim_{r \to \infty} \frac{A_2}{r} = 0 \quad \Rightarrow \quad A_1 = C_{\infty}
\]

\[
c(r) = C_{\infty} \left( 1 - \frac{a}{r} \right)
\]

Flux of particles towards the sphere at radius \( r \):

\[
J(r) = -D \frac{dc}{dr} = -C_{\infty}D \frac{a}{r^2}
\]
Diffusion-limited aggregation

The current of incoming molecules at the surface of the sphere will be:

\[ I(r) = 4\pi a^4 J(r) = -4\pi a D c_\infty = -k_a c_\infty \]

\( k_a \) = rate coefficient

If particles do not disappear, but stick to the surface and become sticky:

Diffusion-limited aggregation (simulation)

Bacterial colony
Diffusion from a point source

Diffusion from a point source gives rise to expanding Gaussians. In one dimension:

\[ \frac{dc}{dt} = D \frac{\partial^2 c}{\partial x^2} \quad \Rightarrow \quad c(x,t) = \frac{n_0}{\sqrt{4\pi D t}} e^{-\frac{x^2}{4D t}} = \frac{n_0 e^{-\frac{(x-\mu)^2}{2\sigma^2}}}{\sqrt{2\pi \sigma^2}} \]

The mean: \( \mu(t) = 0 \)  
The standard deviation: \( \sigma(t) = \sqrt{2Dt} \)

In more dimensions:

\[ c(r,t) = \frac{n_0}{(4\pi D t)^{d/2}} e^{-\frac{r^2}{4Dt}} \]
Smoluchowski equation: Diffusion + Drift

Besides diffusion, a force $f$ can act on particles (e.g., electric field acts on charges).

$$J = -D \frac{\partial c}{\partial x} + \frac{cf}{\xi}$$

$$\frac{dc}{dt} = D \frac{\partial^2 c}{\partial x^2} - \frac{f}{\xi} \frac{\partial c}{\partial x}$$

The solution can describe a horse race:

$$c(x,t) = \frac{n_0}{\sqrt{4\pi Dt}} e^{-\frac{(x-\nu t)^2}{4Dt}}$$

The mean: $\mu(t) = \nu t$

The standard deviation: $\sigma(t) = \sqrt{2Dt}$
Einstein-Smoluchowski equation: Settling

Assume that particles settle under the effect of gravity in solution. After sufficient time diffusion and gravity balance each other at steady state.

\[
\frac{dc}{dt} = D \frac{\partial^2 c}{\partial x^2} - f \frac{\partial c}{\partial x} = 0 \quad \Rightarrow \quad D \frac{dc}{c} = \frac{f}{\xi} \, dx \quad \Rightarrow \quad c(x) = c(0)e^{\frac{\Delta \varepsilon}{\xi D}}
\]

Boltzmann distribution law: \( c(x) = c(0)e^{\frac{\Delta \varepsilon}{kT}} \)

\[
\Rightarrow \quad D = \frac{kT}{\xi}
\]

\( f = mg \)
\( \Delta \varepsilon = mgx \)

Yeast cell cultures
Yeast cells: \( m_1 < m_2 \)
Friction is related to Diffusion

\[ D = \frac{kT}{\xi} \] - Once we know D, we can calculate \( \xi \) and vice versa.

Both depend on the shape and the dimensions of particles, and the viscosity of the medium.

For example, for a sphere:

\[ \xi = 6\pi \eta a \quad \rightarrow \quad D = \frac{kT}{6\pi \eta a} \]
Example: Diffusion coefficients for polymers

Polymers could have various lengths, which determine their weights and 3-dimensional sizes.

\[ D = \frac{kT}{6\pi \eta a} = \frac{kT}{6\pi \eta} \left( \frac{4\pi \rho}{3w_m} \right)^\frac{1}{d} \]

\[ \ln(D) = K - \frac{1}{d} \ln(w_m) \]

1 < d < 3 = dimensionality of the polymer
Sources and sinks: Population biology

Consider a growing population with unlimited food. \[
\frac{dc}{dt} = gc \quad \Rightarrow \quad c(t) = c(0)e^{gt}
\]

More realistically, with finite food \( f \) such that \( f + c = b \)
\[
\frac{dc}{dt} = gfc = gc(b - c)
\]
\[
\frac{dc}{c(b - c)} = gdt \quad \Rightarrow \quad \frac{dc}{c} - \frac{dc}{c - b} = gbdt
\]
\[
\frac{c}{c - b} = Ce^{gbt} \quad \Rightarrow \quad c(t) = \frac{bc(0)e^{bgt}}{b - c(0)[1 - e^{bgt}]}
\]

Verhulst equation for population growth
Fisher model: Growth with diffusion

Now we introduce diffusion besides population growth.

\[ \frac{dc}{dt} = D \nabla^2 c + gc(b - c) \]

This is an example of a reaction-diffusion system.

The solution is an expanding wavefront.

In 1 dimension:

In 2 dimensions:
Other reaction-diffusion systems

The following reaction-diffusion system is the FitzHugh-Nagumo model. It can describe calcium waves in cells, heart dynamics, brain activity or chemical waves.

\[
\begin{align*}
\varepsilon \frac{dv}{dt} &= v(a - v)(v - 1) - w + D \nabla^2 v \\
\frac{dw}{dt} &= v - w - b
\end{align*}
\]

Perturbations can propagate much faster than by diffusion alone!

Waves in the rat brain

Waves in the dog heart

Solutions of FitzHugh-Nagumo equations

- Spiral
- Bull’s eye
- Pulse