Gauss’ Law.
Electrostatic potentials.
Challenge: Particle fluxes

Fluid with concentration $c$ is flowing across two surfaces with the same speed. Which one has more particles flowing through per unit time?

Particles are emitted from a point source at rate $\Delta N/\Delta t$, creating a concentration $c$. How many particles flow through a sphere centered on the source?
A bit of geometry: Areas
Fluid with concentration \( c \) is flowing across two surfaces \( A_0, A_1 \) with the same speed. Which one has more particles flowing through per unit time?

\[
\frac{\Delta N_0}{\Delta t} = cA_0 \frac{\Delta x}{\Delta t} = \frac{\Delta N_1}{\Delta t} = cA_1 \frac{\Delta x}{\Delta t} \cos(\theta) = c\vec{A}_1 \vec{v} = c\vec{A}_1 \vec{v}
\]

Particles are emitted from a point source at rate \( \Delta N/\Delta t \), creating a concentration \( c \). How many particles flow through a sphere centered on the source?

\[
\int_S c\vec{v}\vec{n}dS = \frac{\Delta N}{\Delta t}, \text{ independent on the radius.}
\]
The flux of the electric field

The flux of the electric field through a given surface is given by:

\[ \Phi = \int \mathbf{D} \mathbf{E} \, ds \]

**Example:**
For a sphere with a point charge at its center, \( D=1 \):

\[ \Phi = \int_{\text{surface}} \left( \frac{1}{4\pi\varepsilon_0} \frac{q}{R^2} \right) \mathbf{R} \, ds = \frac{1}{4\pi\varepsilon_0} \frac{q}{R^2} \int_{\text{surface}} ds \]

\[ \Phi = \frac{q}{4\pi R^2 \varepsilon_0} 4\pi R^2 = \frac{q}{\varepsilon_0} \]

The flux is independent of:
- The dielectric constant
- The sphere’s radius...
The flux is independent of the closed surface shape

Imagine an outer balloon surrounding the inner sphere and the charge. For a given surface element:

\[ \Phi_{\text{outer}} = D E(R) S(R) \cos \theta \]

\[ S_{\perp}(R) = S(r) R^2 \]

\[ E(R) = \frac{1}{4\pi\varepsilon_0} \frac{q}{DR^2} = E(r) \frac{r^2}{R^2} \]

\[ \Phi_{\text{outer}} = D E(r) \frac{r^2}{R^2} S(r) R^2 \cos \theta = \Phi_{\text{inner}} \]

For any surrounding surface: \[ \Phi = \frac{q}{\varepsilon_0} \]
Gauss’ Law

For multiple charges surrounded by a surface we obtain Gauss’ Law:

\[ \Phi = \int_{\text{surface}} D E ds = \frac{1}{4\pi\varepsilon_0} \int_{\text{surface}} D(E_1 + E_2 + \ldots) ds = \frac{1}{\varepsilon_0} \sum_i q_i \]

Flux through a closed surface is proportional with the sum of charges in its interior.

For charge that is continuously distributed with charge density \( \rho(x,y,z) \):

\[ \Phi = \int_{\text{surface}} D E ds = \frac{1}{\varepsilon_0} \int_{V} \rho dV \]
Gauss’ Law: Applications
Gauss’ Law: Continuously distributed charge

- in space, charge density $= \rho(x,y,z)$:
  $$ \Phi = \int_S D \mathbf{E} \cdot ds = \frac{1}{\varepsilon_0} \int_V \rho \, dV $$

- on a surface, charge density $= \sigma(x,y)$:
  $$ \Phi = \int_S D \mathbf{E} \cdot ds = \frac{1}{\varepsilon_0} \int_A \sigma \, dA $$

- on a line, charge density $= \lambda(x)$:
  $$ \Phi = \int_S D \mathbf{E} \cdot ds = \frac{1}{\varepsilon_0} \int_A \lambda \, dL $$
The electric field of a line charge

Consider a long line with charge density $\lambda$. Then a segment of length $L$ has charge $= L\lambda$.

What is the electric field around the line? Apply Gauss’ Law for the flux through a closed surface surrounding the line. (Flux is proportional with the sum of charges in its interior...)

$$\Phi = \int_{s} D(E)ds = 2\pi DrE = \int_{V} QdV = \frac{\lambda L}{\varepsilon_0}$$

$$2\pi DrLE = \frac{\lambda L}{\varepsilon_0}$$

$$E = \frac{\lambda}{2\pi\varepsilon_0 Dr}$$

The electric field drops slower than for a point charge.
The electric field of a charged planar surface

Consider a plane with charge density $\sigma$. Then a segment of area $A$ has charge $= A\sigma$. Apply Gauss’ Law to get the flux through a closed cylindrical surface dissected by the plane.

(a) For a thin plane:

$$\Phi = \int_S D E d s = 2 D E A = \int_V Q d V = \frac{\sigma A}{\varepsilon_0} \Rightarrow E = \frac{\sigma}{2 D \varepsilon_0}$$

(b) For a thick plane:

$$\Phi = \int_S D E d s = D E A = \int_V Q d V = \frac{\sigma A}{\varepsilon_0} \Rightarrow E = \frac{\sigma}{D \varepsilon_0}$$

The electric field does not depend on the distance in either case.
How about two oppositely charged planar surfaces?

Consider two infinite planes with charge densities $+\sigma$ and $-\sigma$. Electric fields are additive (as vectors).

Between the two planes the electric field is:

$$E = \frac{\sigma_+ - (-\sigma_-)}{2D\varepsilon_0} = \frac{\sigma}{D\varepsilon_0}$$

Biological example: A neuronal cell membrane
Review: Vector calculus

The “nabla” operator gives the gradient field (steepest ascent) of scalar functions:

\[ \nabla = \frac{\partial}{\partial x} \hat{i} + \frac{\partial}{\partial y} \hat{j} + \frac{\partial}{\partial z} \hat{k} \quad \Rightarrow \quad \nabla f = \frac{\partial f}{\partial x} \hat{i} + \frac{\partial f}{\partial y} \hat{j} + \frac{\partial f}{\partial z} \hat{k} \]

The “Laplacian” operator gives the sources or sinks of gradient vector fields:

\[ \nabla^2 = \Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \quad \Rightarrow \quad \nabla^2 f = \Delta f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} \]

Divergence-less fields have no sources or sinks, so their flux through any closed surface is 0.
Visual example

Contour lines: same height

Arrows: gradient

Sink: arrows vanish
Electrostatic potential, benzene molecule
Work needed to move a charge

The electrostatic force acting on charge $q$ in the electric field of another charge $Q$ is:

$$ f(r) = q \frac{Q}{4\pi \varepsilon_0 Dr^2} \frac{r}{r} = qE(r) $$

Now we use a force opposing $f$ to move the charge $q$ very slowly over a distance $dl$.

The work done will be:

$$ \delta W = f_{ext}(r) \cdot dl = -qE(r) \cdot dl $$

If we move the charge $q$ from point A to point B:

$$ W_{AB} = -q \int_A^B \mathbf{E}(r) \cdot d\mathbf{l} = -q \int_A^B \frac{Q}{4\pi \varepsilon_0 Dr^2} \, dr $$

Example:

$$ \delta W = -\frac{qQdr}{4\pi \varepsilon_0 Dr^2} $$
Electrostatic potential: Work done on unit charge

Electric potential difference between A and B = work needed to move unit charge from A to B:

\[ \psi_B - \psi_A = \frac{W_{AB}}{q} = -\int_A^B \mathbf{E}(\mathbf{r}) \cdot d\mathbf{l} \]

The electric potential of point B is the work needed to move a unit charge from infinity to B:

\[ \psi_B = -\int_{\infty}^B \mathbf{E}(\mathbf{r}) \cdot d\mathbf{l} = \int_B^{\infty} \mathbf{E}(\mathbf{r}) \cdot d\mathbf{l} \]

Example:

\[ \psi_{AB} = \frac{Q}{4\pi\varepsilon_0 D} \left[ \frac{1}{r_B} - \frac{1}{r_A} \right] \]
Electrostatic potential around a point charge

Potential of a point charge $Q = \text{work needed to bring a test charge from far away to a given } r$.

$$\psi(r) = -\int_{\infty}^{r} E dr' = -\frac{Q}{4\pi\varepsilon_0 D} \int_{\infty}^{r} \frac{1}{r'^2} dr' = \frac{Q}{4\pi\varepsilon_0 D r}$$

For continuous charge density:

$$\psi(r) = \frac{1}{4\pi\varepsilon_0 D} \int_{V} \frac{\rho}{r'^2} dV$$
Electric field is = Gradient of the potential

For a point charge $q$ the potential is:

$$\psi(r) = \frac{q}{4\pi\varepsilon_0 Dr} = -\int E(r)dr$$

The corresponding electric field is:

$$E(r) = \frac{q}{4\pi\varepsilon_0 Dr^2} = -\frac{\partial \psi}{\partial r}$$

In general we have: $\mathbf{E}(\mathbf{r}) = -\nabla \psi$

This is similar to the force being the gradient of potential energy: $\mathbf{f}(\mathbf{r}) = -\nabla u$
Electrostatic potential surfaces

Potential surfaces are regions of space where the potential does not change.

For a point charge: \[ \psi(r) = \frac{q}{4\pi\varepsilon_0 Dr} = \text{const} \]

This means potential surfaces are concentric spheres.

Analogy: geographic contour lines (equal height). Work to drag a mass uphill: \( w = mg\Delta h \)
Electric field is perpendicular to potential surfaces

Gradient points towards steepest increase in potential. The steepest increase is perpendicular to potential (just like heading straight uphill). Therefore, the electrostatic field vector is perpendicular to equipotential surfaces.

Conductors are equipotential volumes and surfaces. Otherwise the resulting electric field would cause movement of charges until the potential becomes constant.
Examples of equipotential surfaces

Two positive charges

A protein surface
Electric interactions are conservative forces

Work necessary to move a charge \( q \) can be decomposed into steps that are either **perpendicular** or **tangential** to equipotential surfaces.

Work due to **tangential** movement is \( = 0 \).

Work due to **perpendicular** movement alters the equipotential surface and alters energy.

\[
\begin{align*}
w_{AC} = w_{BC} &= -q_2 \int_{B}^{C} \frac{q_1}{4\pi\varepsilon_0 D r^2} dr = w_{AD} \\
\end{align*}
\]

\[
\begin{align*}
w_{AC} &= q_2 \frac{1}{4\pi\varepsilon_0 D} \left( \frac{1}{r_C} - \frac{1}{r_B} \right) = q_2 [\psi_C (r) - \psi_A (r)]
\end{align*}
\]

The work and potential are path-independent. Only the initial and final potential surfaces matter.
Electrostatic potential of a parallel-plate capacitor

Electrostatic potentials are additive.

At any point between two large charged planes, the electric field is:

\[ E(x) = \frac{\sigma}{2\varepsilon_0 D} + \frac{\sigma}{2\varepsilon_0 D} = \frac{\sigma}{\varepsilon_0 D} \]

\[ \Delta \psi = -\int_0^x \frac{\sigma}{\varepsilon_0 D} dx' = -\frac{\sigma \Delta x}{\varepsilon_0 D} \]

Two charged planes = capacitor.
Capacitors store charge.
Capacitance = charge per potential difference.

\[ C = \frac{\sigma A}{|\Delta \psi|} = \frac{\varepsilon_0 DA}{d} \]
Capacitors in biology

The nerve cell stores charge and maintains a potential difference.

Capacitance for unit length of axon (lipid bilayer):

\[
C = \frac{\varepsilon_0 DA}{d} = \frac{2\pi R L \varepsilon_0 D}{dL} = 1.112 \text{ \(\mu\text{F/m}\)}
\]
How do sharks know where is prey?

Sharks can detect electric fields of one millionth of a volt per centimeter.

Dip poles of 1.5 V battery into:
- Long Island Sound
- Jacksonville, FL

A shark can detect it! How?
Ampullae of Lorenzini

The shark applies the potential difference between its body and water onto a cell membrane $\sim 10\,\text{Å}$ thick.

Amplification $\sim 10^9$-fold.

$$E_1 = \frac{V_{\text{body}} - V_{\text{water}}}{d_{\text{shark}}}$$

$$E_2 = \frac{V_{\text{body}} - V_{\text{water}}}{d_{\text{membrane}}}$$