Electrostatic potentials

Monday, September 21, 2015
SBU:CHE/PHY558, Physical & Quantitative Biology
Lecturer: Gábor Balázsi
Electric field of a planar surface

Consider a uniformly charged plane (charge density=$\sigma$). We want to know the electric field. Take a cylinder with faces perpendicular to the plane.

\[ 2DAE(x) = \frac{A\sigma}{\varepsilon_0} \]
\[ E(x) = \frac{\sigma}{2\varepsilon_0 D} \]

(a) Thin Plane

\[ DAE(x) = \frac{A\sigma}{\varepsilon_0} \]
\[ E(x) = \frac{\sigma}{\varepsilon_0 D} \]

(b) Thick Plane
Electrostatic potential

Assume that a charge $q$ is placed in the electric field of another charge $Q$. The force acting on charge $q$ is:

$$f(r) = q \frac{Q}{4\pi\varepsilon_0 D r^2} \frac{r}{r} = qE(r)$$

Now we use a force opposing $f$ to move the charge $q$ very slowly over a distance $dl$. The work done will be:

$$\delta W = f_{ext}(r) \cdot dl = -qE(r) \cdot dl$$
Electrostatic potential

If we move the charge $q$ from point $A$ to point $B$:

$$ w_{AB} = -q \int_A^B \mathbf{E}(\mathbf{r}) \cdot d\mathbf{l} $$

The electric potential difference between $A$ and $B$ is the work needed to move a unit charge:

$$ \psi_B - \psi_A = \frac{w_{AB}}{q} = -\int_A^B \mathbf{E}(\mathbf{r}) \cdot d\mathbf{l} $$

If point $A$ is moved to infinity, then $\psi(x,y,z)$ describes the space.
Electrostatic potential around a point charge

What is the potential of a point charge $q$? The question refers to the entire space. We calculate the work needed to bring a test charge from far away to distance $r$. The electric field is radial, so we don’t need vector notation.

$$\psi(r) = -\int_{\infty}^{r} Edr' = -\frac{q}{4\pi\varepsilon_0 D} \int_{\infty}^{r} \frac{1}{r'^2} dr' = \frac{q}{4\pi\varepsilon_0 D r}$$

For continuous charge density: $$\psi(r) = \frac{1}{4\pi\varepsilon_0 D} \int_{\infty}^{r} \frac{\rho}{r'^2} dV$$

Positive Charge

![Positive Charge Graph](image)

Negative Charge

![Negative Charge Graph](image)
Electric field is the gradient of the potential

For a point charge $q$ the potential is:

$$\psi(r) = \frac{q}{4\pi \varepsilon_0 r}$$

The corresponding electric field is:

$$E(r) = \frac{q}{4\pi \varepsilon_0 r^2} = -\frac{\partial \psi}{\partial r}$$

In general we have: $E(r) = -\nabla \psi$

This is similar to the force being the gradient of potential energy: $f(r) = -\nabla u$
Electrostatic potential surfaces

Potential surfaces are regions of space where the potential does not change.

For a point charge: \( \psi(r) = \frac{q}{4\pi\varepsilon_0 Dr} = \text{const} \)

This means potential surfaces are concentric spheres.

Analogy: geographic contour lines (equal height). Work to drag a mass uphill: \( w = mg\Delta h \)
Electric field is perpendicular to potential surfaces

Gradient points towards steepest increase in potential. The steepest increase is perpendicular to potential (just like heading straight uphill). Therefore, the electrostatic field vector is perpendicular to equipotential surfaces.

Conductors are equipotential volumes and surfaces. Otherwise a non-perpendicular field component would cause movement of charge until the potential becomes constant.
Electric interactions are conservative forces

Work necessary to move a charge $q$ can be decomposed into multiple steps that are either **perpendicular** or **tangential** to equipotential surfaces.

Work due to **tangential** movement is $= 0$.

Work due to **perpendicular** movement alters the equipotential surface and energy.

$$w_{AC} = w_{BC} = -q_2 \int_B^C \frac{q_1}{4\pi\varepsilon_0 D r^2} dr$$

$$w_{AC} = \frac{q_2}{4\pi\varepsilon_0 D} \left( \frac{1}{r_C} - \frac{1}{r_B} \right) = q_2 \left[ \psi_C(r) - \psi_A(r) \right]$$

The path of movement does not matter. Only the initial and final potential surfaces matter.
Image charges: a charge above a conducting plane

What is the field of a charge over a conducting plane?

Conductor surface is equipotential. The electric field must be perpendicular to it.

The system is equivalent to having an opposite charge below the surface.
Equipotential surfaces of a dipole

(a) Dipole Field
Other examples of equipotential surfaces

Two positive charges

A protein surface
Electrostatic potential of a parallel-plate capacitor

What is the work needed to move a unit charge between the plates?

At any point far from the edges, the electric field is:

\[ E(x) = \frac{\sigma}{2\varepsilon_0 D} + \frac{\sigma}{2\varepsilon_0 D} = \frac{\sigma}{\varepsilon_0 D} \]

\[ \Delta \psi = -\frac{\sigma\Delta x}{\varepsilon_0 D} \]

This is a capacitor. Capacitors store charge. Capacitance is charge per potential difference.

\[ C = \frac{\sigma A}{\Delta \psi} = \frac{\varepsilon_0 DA}{d} \]
Capacitors in electronics and biology

Nerve cell

The nerve cell stores charge and maintains a potential difference.

Capacitor (electronics)